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Infinite Product Representations for Kernels and Iteration of Functions

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INFINITE PRODUCT REPRESENTATIONS FOR KERNELS AND ITERATIONS OF FUNCTIONS

DANIEL ALPAY, PALLE JORGENSEN, IZCHAK LEWKOWICZ,
AND ITZIK MARTZIANO

Dedicated to the memory of Bill Arveson

ABSTRACT. We study infinite products of reproducing kernels with view to their use in dynamics (of iterated function systems), in harmonic analysis, and in stochastic processes. On the way, we construct a new family of representations of the Cuntz relations. Then, using these representations we associate a fixed filled Julia set with a Hilbert space. This is based on analysis and conformal geometry of a fixed rational mapping R in one complex variable, and its iterations.

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1. INTRODUCTION

The purpose of this paper is twofold, one is to offer a general framework for an harmonic analysis of reproducing kernel Hilbert spaces, and the other is to explore its applications. On the first point, we offer a general tool for analysis of positive definite kernels, and their associated reproducing kernel Hilbert spaces (RKHS). Our analysis is based on a construction of families of representations of a system of operators. We use these representations in order to construct explicit factorizations of the kernels at hand; and, as a result we get computable decompositions of the associated reproducing kernel Hilbert spaces. Our starting point is a symbolic set with relations, often called the Cuntz relations (CR) after the C^* -algebra they generate, see [10]. The CRs are indexed by the number n of symbols in a generating system; for every n , including possibly $n = \infty$, we have a Cuntz algebra \mathcal{O}_n , see [10].

Our present results were announced in [3]. While representations of the Cuntz algebras \mathcal{O}_n , denoted $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ for a given separable Hilbert space \mathcal{H} , have a long history in operator algebras and their applications, our present use of them in an analysis of dynamics of complex substitution-systems is novel. The use of $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ in operator algebras and physics was pioneered by Arveson [5]; see also [8], and the references there. Fix n , then, up to a natural action of the group U_n , the space $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ is $\text{End}_n(\mathbf{B}(\mathcal{H}))$ where $\mathbf{B}(\mathcal{H})$ denotes the set of all bounded operators on \mathcal{H} and End_n denotes the set of all endomorphisms of $\mathbf{B}(\mathcal{H})$ of Powers index n .

1.1 Representations. Our present use of the CRs centers on an analysis of representations of \mathcal{O}_n ; so our focus is on the representations as opposed to C^* -algebraic questions. By a representations of the CRs we mean a multivariable operator system which satisfies the formal Cuntz relations. The study of representations of the Cuntz relations is subtle for a number of reasons. For example, it is known (see e.g., [7]) that, in general, for n fixed, that the variety of all representations of \mathcal{O}_n (up to unitary equivalence) is very large; to be precise, in fact the set of equivalence classes of irreducible representations of \mathcal{O}_n is known to not admit a measurable parameterization with any Borel set. Nonetheless, for diverse purposes, there are interesting infinite dimensional families of representations serving diverse purposes in harmonic analysis and in

applications. Our focus here is a set of applications of representations to an harmonic analysis of kernel Hilbert spaces. With this, we then arrive at decompositions which are of relevance in the study of sub-band filters in signal processing, in an harmonic analysis of de Branges spaces, and of use in building \mathbf{L}_2 -spaces on fractals, and in an analysis on Hilbert spaces built on Julia sets arising in conformal geometry from iteration of fixed rational mappings.

1.2 Kernels. The decomposition of a positive definite kernel into a sum of such kernels is not unusual, but the decomposition of a positive definite kernel into a product of positive definite kernels seems much less common. In the work [2], such a multiplicative decomposition was used in the setting of the Hardy space of the open unit disk, and connections with the Cuntz relations were pointed out. In the present work, we study positive kernels of the form

$$K(z, w) = \prod_{n=0}^N k(R_n(z), R_n(w)),$$

and

$$K(z, w) = \prod_{n=0}^{\infty} k(R_n(z), R_n(w)).$$

In these expressions, $k(z, w)$ is a positive definite function on a set \mathcal{U} , and R is a map from \mathcal{U} into itself. We denote by $R_0(z) = z$ and

$$(1.1) \quad R_n = \underbrace{R \circ R \circ \cdots \circ R}_{n \text{ iterations}},$$

and appropriate assumptions are made to insure convergence.

1.3 Local analysis. Our paper is at the crossroads of complex dynamics and representation theory (representations of certain non-abelian algebras). To aid readers from both areas, we have included some preliminaries on the two subjects. Beginning with complex dynamics, recall that Julia sets (and their compliments Fatou sets $F(f)$) serve to chart a number of geometric patterns of points under iterated substitution. Specifically, the two complementary sets are defined from a given (and fixed) function f ; often f is a rational function defined on the Riemann sphere. Intuitively, the Fatou set of f is the set of points such that all nearby values under f behave similarly under repeated iteration (self-substitution): $F(f)$ is open, and the iterations form a normal family in $F(f)$ in the sense of Montel. By contrast, the points

in the compliment, the Julia set $J(f)$ have the property that an arbitrarily small perturbation can cause drastic changes in the sequence of iterated function values, i.e., points in $J(f)$ represents chaotic behavior. The bounded connected component of $F(f)$ is called the filled Julia set, and $J(f)$ is its boundary. In these dynamical terms, the intuition is that points in the Fatou set are 'regular', and the Julia set represents 'chaos'. In terms of geometry of repeated iteration, the Julia set falls in the realm between deterministic and stochastic. As a result, the Julia set is aperiodic, and so does not lend itself to standard tools of harmonic analysis. Our aim is to zoom in on local properties of points in the filled Julia set. It is our aim to show, for certain examples (some cases when f is a polynomial), that there is a local version of a harmonic analysis which works well in a neighborhood of attracting cycles in the filled Julia set, or in an open basin of attraction for an attractive fixed point of f . Due to a theorem of Brolin [9], every given f has an invariant maximal-entropy measure μ (depending on f), and the support of μ is $J(f)$. Intuitively, in our analysis, μ plays a role analogous to that played by Haar measure in the harmonic analysis of compact groups. A key tool in our analysis is the design of representations of a system of isometries in the Hilbert space $H := \mathbf{L}_2(\mu)$ defined from $\mu = \mu(f)$. Such a system of N isometries was defined axiomatically by J. Cuntz (for entirely different purposes), and when N is fixed we speak of the Cuntz algebra O_N .

In general, the theory of the representations of O_N is difficult. Nonetheless, for our present purpose, we show that there is a distinguished subclass of representations, important for our local harmonic analysis in the filled Julia set. To help understand this, note that from every representation of O_N acting on a fixed Hilbert space H , one naturally obtains an associated N -nary system of mutually orthogonal subspaces in H . This is a Hilbert space theoretic tree-like symbolic representation of local features of the dynamics. We will construct our representations such that the corresponding N -nary system in $H = \mathbf{L}_2(\mu)$ corresponds to the geometry of iterated backward substitutions for a fixed polynomial f .

2. PRELIMINARIES ON REPRODUCING KERNEL HILBERT SPACES

In this section we review some facts on positive definite functions and on operators between reproducing kernel Hilbert spaces. See for instance [4, 21]. The knowledgeable reader can proceed directly to Section 3. We divide the section into two subsections, devoted to definitions, and $\mathcal{L}(\varphi)$ de Branges spaces.

2.1. Generalities. First recall that a function $K(z, w)$ defined on a set Ω is positive definite if for every choice of $M \in \mathbb{N}$, of points $w_1, \dots, w_M \in \Omega$ and $c_1, \dots, c_M \in \mathbb{C}$, it holds that

$$\sum_{k, \ell=1}^M c_\ell^* K(w_\ell, w_k) c_k \geq 0.$$

Associated to a positive definite function is a unique Hilbert space $\mathcal{H}(K)$ of functions on Ω , with the following two properties: For every $w \in \Omega$, the function $K_w : z \mapsto K(z, w)$ belongs to $\mathcal{H}(K)$, and for every $f \in \mathcal{H}(K)$,

$$\langle f, K_w \rangle_{\mathcal{H}(K)} = f(w).$$

The function K is called the reproducing kernel of the space and it can be computed by the formula

$$(2.1) \quad K(z, w) = \sum_{j \in J} e_j(z) e_j(w)^*,$$

where $(e_j)_{j \in J}$ is any orthonormal basis of $\mathcal{H}(K)$. Note that J need not be countable since the space $\mathcal{H}(K)$ need not be separable.

In the case of the Hardy space $\mathbf{H}_2(\mathbb{C}_+)$ of the open right-half-plane \mathbb{C}_+ , an orthonormal basis for $\mathbf{H}_2(\mathbb{C}_+)$ is given by the functions

$$t_n(z) = \frac{1}{\sqrt{\pi}} \frac{1}{z+1} \left(\frac{z-1}{z+1} \right)^n, \quad n = 0, 1, \dots$$

and we have

$$\frac{1}{2\pi(z+w^*)} = \sum_{n=0}^{\infty} t_n(z) t_n(w)^*.$$

Proposition 2.1. *Let K be positive definite on Ω and let φ denote a function from Ω into itself, and let e be a function from Ω into \mathbb{C} . The operator*

$$Sf(z) = e(z)f(\varphi(z))$$

is bounded from $\mathcal{H}(K)$ into itself if and only if the function

$$(2.2) \quad K(z, w) - e(z)e(w)^* K(\varphi(z), \varphi(w))$$

is positive on Ω . When this condition holds, the adjoint operator is given by the formula

$$(2.3) \quad S^* K_w = e(w)^* K_{\varphi(w)}.$$

While the Proposition can be found in the literature, we include below a sketch of the main idea involved as it serves to unify several themes coming later in a variety of seemingly different context; harmonic analysis, representation theory, the study of Julia sets, to mention a few. The details below further serve to introduce terminology to be used later. To prove the proposition consider the linear relation in $\mathcal{H}(K) \times \mathcal{H}(K)$ spanned by the pairs

$$(K(\cdot, w), e(w)^* K(\cdot, \varphi(w))), \quad w \in \Omega.$$

It is densely defined. Moreover it is contractive thanks to (2.2). Therefore it extends to the graph of an everywhere defined contraction, say T . For $f \in \mathcal{H}(K)$ and $w \in \Omega$ we have

$$\begin{aligned} \langle T^* f, K(\cdot, w) \rangle_{\mathcal{H}(K)} &= \langle f, T(K(\cdot, w)) \rangle_{\mathcal{H}(K)} \\ &= \langle f, e(w)^* K(\cdot, \varphi(w)) \rangle_{\mathcal{H}(K)} \\ &= e(w) f(\varphi(w)) \\ &= (Sf)(w). \end{aligned}$$

2.2. $\mathcal{L}(\varphi)$ spaces. Recall that a $\mathbb{C}^{n \times n}$ -valued function φ is analytic in \mathbb{C}_+ and such that $\operatorname{Re} \varphi(z) \geq 0$ for $z \in \mathbb{C}_+$ if and only if it is of the form

$$(2.4) \quad \varphi(z) = a + bz - i \int_{\mathbb{R}} d\mu(t) \left\{ \frac{1}{t - iz} - \frac{t}{t^2 + 1} \right\},$$

where $a \in \mathbb{C}^{n \times n}$ is such that $a + a^* = 0$, $b \in \mathbb{C}^{n \times n}$ is non negative, and where $d\mu$ is a $\mathbb{C}^{n \times n}$ -valued positive measure subject to

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + 1} < \infty$$

This expression allows to extend φ to the open left half plane (the extension will not be continuous across the imaginary axis in general). When φ is extended in such a way, the kernel

$$(2.5) \quad \frac{\varphi(z) + \varphi(w)^*}{z + w^*} = b + \int_{\mathbb{R}} \frac{d\mu(t)}{(t - iz)(t - iw)^*}$$

is positive definite in $\mathbb{C} \setminus i\mathbb{R}$. The associated reproducing kernel Hilbert space $\mathcal{L}(\varphi)$ was characterized and studied by de Branges. It consists of functions of the form

$$F(z) = b\xi + \int_{\mathbb{R}} \frac{d\mu(t)f(t)}{t - iz},$$

with $\xi \in \mathbb{C}^n$ and $f \in \mathbf{L}_2(d\mu)$, with norm

$$\|F\|^2 = \xi^* b \xi + \|f\|_\mu^2.$$

See [11], [1]. The space is finite dimensional if and only if μ is a jump measure with a finite number of jumps. This happens if and only if the function ϕ is rational and satisfies

$$\varphi(z) = -\varphi(-z^*)^*, \quad \forall z \in \mathbb{C}, \text{ which is not a pole of } \phi.$$

We now consider $n = 1$ and consider a rational function φ such that $\mathcal{L}(\varphi)$ is finite dimensional. Thus

$$\varphi(z) = a + bz + \sum_{j=1}^N \frac{m_j}{t_j - iz}$$

where the $t_j \in \mathbb{R}$ and the $m_j > 0$.

Proposition 2.2. *Let $a = b = 0$. Let*

$$e_j(z) = \frac{\sqrt{m_j}}{(t_j - iz)}, \quad j = 1, \dots, N.$$

Then,

$$(2.6) \quad \frac{\varphi(z) + \varphi(w)^*}{z + w^*} = \sum_{n=1}^N e_n(z) e_n(w)^*,$$

and

$$(2.7) \quad \langle e_j, e_k \rangle_{\mathcal{L}(\phi)} = \delta_{jk}.$$

Proof: Equation (2.6) is a special case of (2.5) when μ is a jump measure with a finite number of jumps. Formula (2.7) is a special case of (2.1) since the e_j are linearly independent. \square

3. A GENERAL SETTING

We now assume that the set \mathcal{U} in the introduction is a topological space and obtain infinite product representations of certain positive definite kernels. We start from a function $k(z, w)$ positive definite in \mathcal{U} and denote by $(e_j)_{j \in J}$ denote an orthonormal basis of the reproducing kernel Hilbert space $\mathcal{H}(k)$ with reproducing kernel $k(z, w)$ (with $z, w \in \mathcal{U}$). Thus,

$$(3.8) \quad k(z, w) = \sum_{j \in J} e_j(z) (e_j(w))^*, \quad z, w \in \mathcal{U}.$$

At this stage, $\mathcal{H}(k)$ need not be separable, and thus the index set J need not be countable. We assume that \mathcal{U} satisfies

$$(3.9) \quad K(z, w) = k(z, w)K(R(z), R(w)), \quad \forall z, w \in \mathcal{U}.$$

where the function $K(z, w)$ is not *a priori* not positive definite in \mathcal{U} .

Proposition 3.1. *Let $k(z, w)$ positive definite in \mathcal{U} , and assume that (3.9) is in force for some function $K(z, w)$, which is continuous on \mathcal{U} , and not identically equal to 0. Assume that there exists a point $\ell \in \mathcal{U}$ such that*

$$(3.10) \quad \lim_{n \rightarrow \infty} R_n(z) = \ell, \quad \forall z \in \mathcal{U}.$$

Then if $K(\ell, \ell) > 0$, the function $K(z, w)$ is positive definite in \mathcal{U} and

$$K(z, w) = \left(\prod_{n=0}^{\infty} \left(\sum_{i \in I} e_i(R_n(z)) e_i(R_n(w))^* \right) \right) K(\ell, \ell).$$

Proof: Let $N \in \mathbb{N}_0$. It holds that

$$(3.11) \quad K(z, w) = \left(\prod_{n=0}^N k(R_n(z), R_n(w)) \right) K(R_{N+1}(z), R_{N+1}(w)),$$

for $z, w \in \mathcal{U}$. We note that $R(\ell) = \ell$. The hypothesis imply that

$$\lim_{N \rightarrow \infty} K(R_{N+1}(z), R_{N+1}(w)) = K(\ell, \ell) > 0,$$

and hence the infinite product $\prod_{n=0}^{\infty} k(R_n(z), R_n(w))$ converges for $z, w \in \mathcal{U}$ and is equal to $\frac{K(z, w)}{K(\ell, \ell)}$. \square

In the preceding proposition, one does not assume that $k(z, w) - 1$ is positive definite in \mathcal{U} . As examples, we mention the works [19], [], where positive definite kernels of the form

$$\prod_{n=0}^{\infty} \cos \frac{(t-s)}{4^n}$$

are introduced. The main point of the example is to illustrate that our method works in examples that have more gaps than are usually involved in the standard theory for inverse iteration of branches of some fixed polynomial, or rational function $R(z)$. Starting with R of degree N , say, it is natural to create an IFS corresponding to a choice of N branches of inverse for R . In more detail, let R be fixed, and let the degree of R be N . Then the Riemann surface for R has N sheets, and R will be onto with a system of N functions, serving as branches of inverse for R . One then iterates such an N -nary system of inverses (see [13].)

Even choosing for R just the monomial $R(z) := z^4$ leads to IFSs with gaps of interest in harmonic analysis of lacunary Fourier expansions. Recall that in the gap-examples, such as $R(z) = z^4$, initially there are four distinct functions as inverse for R , but one may select only two of them for an IFS. The result is a fractal with gaps, and dimension $1/2$. It can be represented as a Cantor set $J(4, 2)$, here realized as a subset of the circle (= the Julia set for R). In this case, the Brolin measure μ (see [9]) coincides with the IFS measure (of dimension $1/2$) corresponding to the choice of two branches of inverse for $R(z) = z^4$. The support of μ coincides with Cantor set $J(4, 2)$. We get a Hilbert space of lacunary power series with $\mathbf{L}_2(\mu)$ -boundary values supported on the Cantor set $J(4, 2)$. We also refer to [16, 15, 12, 14, 18, 17] for related works on IFS and CR.

3.1. A representation of the Cuntz algebra \mathcal{O}_N . In this subsection we construct representations of the Cuntz relations. See [17] for more on these relations.

Theorem 3.2. *Let $\mathcal{H}(K)$ denote the reproducing kernel of functions defined on \mathcal{U} and with reproducing kernel $K(z, w)$. The operators*

$$(S_j f)(z) = e_j(z)f(R(z)), \quad j \in J,$$

are continuous from $\mathcal{H}(K)$ into itself and satisfy

$$\sum_{j \in J} S_j S_j^* = I_{\mathcal{H}(K)}.$$

Proof: Let $w \in \mathcal{U}$. From formula (2.3) we have

$$S_j^* K_w = (e_j(w))^* K_{R(w)},$$

and so

$$(S_j S_j^* K_w)(z) = e_j(z)(e_j(w))^* K(R(z), R(w)).$$

It follows that

$$\begin{aligned} \left(\sum_{j \in J} S_j S_j^* K_w \right)(z) &= \left(\sum_{j \in J} e_j(z)(e_j(w))^* \right) K(R(z), R(w)) \\ &= k(z, w) K(R(z), R(w)) \quad (\text{using (3.8)}) \\ &= K(z, w) \quad (\text{using (5.3)}). \end{aligned}$$

□

Two cases of interest, which will be elaborated upon in the following section, correspond to $\mathcal{U} = \mathbb{D}$, $\varphi(z) = z^2$ and $k(z, w) = 1 + zw^*$ and $k(z, w) = (1 + zw^*)^2$ respectively. In the first case, $K(z, w) = \frac{1}{1 - zw^*}$

and $\mathcal{H}(K)$ is equal to $\mathbf{H}_2(\mathbb{D})$, the Hardy space of the open unit disk. In the second case, $K(z, w) = \frac{1}{(1-zw^*)^2}$ and $\mathcal{H}(K)$ is equal to $\mathbf{B}_2(\mathbb{D})$, the Bergman space of the open unit disk. In the first case the S_j satisfy the Cuntz relations while they do not satisfy these relations in the second case.

4. HARMONIC ANALYSIS OF KERNELS FROM BLASCHKE PRODUCTS AND BERGMANN SPACE

4.1. $\mathcal{H}(b)$ spaces. We set

$$(4.1) \quad b(z) = \prod_{i=1}^N \frac{z - w_i}{1 - zw_i^*}$$

to be a finite Blaschke product of the open unit disk \mathbb{D} . Writing

$$\frac{1}{1 - zw^*} = \frac{1}{1 - b(z)b(w)^*} \frac{1 - b(z)b(w)^*}{1 - zw^*},$$

one obtains a multiplicative decomposition of the Cauchy kernel. Setting e_1, \dots, e_N to be an orthonormal basis of $\mathbf{H}_2 \ominus b\mathbf{H}_2$ we have

$$(4.2) \quad \frac{1}{1 - zw^*} = \left(\sum_{i=1}^N e_i(z)e_i(w)^* \right) \frac{1}{1 - b(z)b(w)^*},$$

and so, for every $M \in \mathbb{N}$,

$$(4.3) \quad \frac{1}{1 - zw^*} = \left(\prod_{n=0}^M \left(\sum_{i=1}^N e_i(b^{\circ n}(z))e_i(b^{\circ n}(w)^*) \right) \right) \frac{1}{1 - b^{\circ(M+1)}(z)(b^{\circ(M+1)}(w))^*}.$$

Assume that

$$w_1 = w_2 = 0$$

in (4.1). Then,

$$(4.4) \quad \lim_{M \rightarrow \infty} b^{\circ M}(z) = 0, \quad \forall z \in \mathbb{D},$$

and we obtain the infinite product representation

$$(4.5) \quad \frac{1}{1 - zw^*} = \prod_{n=0}^{\infty} \left(\sum_{i=1}^N e_i(b^{\circ n}(z))e_i(b^{\circ n}(w)^*) \right),$$

where we have denoted

$$b^{\circ n}(z) = \begin{cases} z, & \text{if } n = 0, \\ \underbrace{(b \circ b \circ \dots \circ b)(z)}_{n \text{ times}}, & \text{if } n = 1, 2, \dots \end{cases}$$

From (4.5) one obtains multiplicative decompositions for the kernels $\frac{1}{(1-zw^*)^t}$, $t = 2, 3, \dots$

Furthermore, (4.5) implies the orthogonal decomposition

$$\mathbf{H}_2 = \oplus_{i=1}^N e_i(\mathbf{H}_2 \ominus b\mathbf{H}_2)$$

and the maps $S_i f(z) = e_i(z)f(b(z))$ are bounded from \mathbf{H}_2 into itself and satisfy the Cuntz relations.

4.2. The Bergmann space. Let b be the Blaschke product of degree N defined in (4.1). In the case of the Bergmann space we have

$$\frac{1}{(1-zw^*)^2} = \frac{1}{(1-b(z)b(w)^*)^2} \frac{(1-b(z)b(w)^*)^2}{(1-zw^*)^2}.$$

Both the kernels

$$\frac{1}{(1-b(z)b(w)^*)^2} \quad \text{and} \quad \frac{(1-b(z)b(w)^*)^2}{(1-zw^*)^2}$$

are positive definite in \mathbb{D} . Furthermore, with $e_i, i = 1, \dots, N$, being an orthonormal basis of $\mathcal{H}(b)$ we have

$$\frac{1}{(1-zw^*)^2} = \sum_{i,j=1}^N e_i(z)e_j(z)e_i(w)^*e_j(w)^* \frac{1}{(1-b(z)b(w)^*)^2}$$

which leads to the decomposition

$$\mathcal{B} = \sum_{i,j=1}^N e_i e_j \mathcal{B}(b).$$

This decomposition will not be orthogonal in general.

The case $b(z) = z^2$ is of special interest. Then,

$$\begin{aligned} \frac{1}{(1-zw^*)^2} &= \frac{(1-b(z)b(w)^*)^2}{(1-zw^*)^2} \frac{1}{(1-b(z)b(w)^*)^2} \\ &= (1+2zw^*+z^2(w^*)^2)K(b(z),b(w)), \end{aligned}$$

and we obtain the multiplicative representation of the Bergmann kernel

$$\frac{1}{(1-zw^*)^2} = \prod_{n=0}^{\infty} (1+2z^{2^n}(w^*)^{2^n}+z^{2^{n+1}}(w^*)^{2^{n+1}})$$

4.3. Functions with real positive part. Furthermore,

$$\begin{aligned}
 (4.6) \quad \frac{I_n}{z + w^*} &= \frac{1}{\varphi(z) + \varphi(w)^*} \cdot \frac{\varphi(z) + \varphi(w)^*}{z + w^*} I_n \\
 &= \sum_{n=1}^N e_n(z) \frac{I_n}{\varphi(z) + \varphi(w)^*} e_n(w)^* \\
 &= \sum_{n=1}^N \sum_{m=0}^{\infty} e_n(z) t_m(\varphi(z)) t_m(\varphi(w))^* e_n(w)^*.
 \end{aligned}$$

Each of the term

$$e_n(z) t_m(\varphi(z)) t_m(\varphi(w))^* e_n(w)^*$$

is a positive definite function, of rank 1. The associated one-dimensional reproducing kernel Hilbert space is spanned by the function

$$z \mapsto e_n(z) t_m(\varphi(z)).$$

These spaces do not intersect since for $(n_1, m_1) \neq (n_2, m_2)$

$$(4.7) \quad a e_{n_1}(z) t_{m_1}(z) + b e_{n_2}(z) t_{m_2}(\varphi(z)) \equiv 0 \implies a = b = 0.$$

By $(\mathbf{H}_2(\mathbb{C}_+))(\varphi)$ the reproducing kernel Hilbert space with reproducing kernel $\frac{1}{\varphi(z) + \varphi(w)^*}$.

Proposition 4.1. $f \in \mathbf{H}_2(\mathbb{C}_+)(\varphi)$ if and only if it can be written as

$$(4.8) \quad f(z) = h(\varphi(z)), \quad h \in \mathbf{H}_2(\mathbb{C}_r)$$

with norm

$$(4.9) \quad \|f\| = \|h\|.$$

Proof: We first note that (4.9) indeed defines a quadratic norm on the linear span of functions of the form (4.8), and makes this span into a Hilbert space. Let $k_w(z) = \frac{1}{z + w^*}$. Then for $h \in \mathbf{H}_2(\mathbb{C}_r)$ and $f = h \circ \varphi$ we have:

$$\begin{aligned}
 \langle f, k_{\varphi(w)^*}(\varphi) \rangle &= \langle h, k_{\varphi(w)^*} \rangle_{\mathbf{H}_2} \\
 &= h(\varphi(w)) \\
 &= f(w).
 \end{aligned}$$

The result follows from the uniqueness of the reproducing kernel Hilbert space associated to a given positive definite function. \square

Proposition 4.2.

$$(4.10) \quad \mathbf{H}_2(\mathbb{C}_+) = \bigoplus_{n=1}^N e_n(\mathbf{H}_2(\mathbb{C}_+))(\varphi).$$

Proof: That the sum (4.10) is indeed orthogonal follows from (4.7). Furthermore, let $f \in \mathbf{H}_2(\mathbb{C}_+)$ be a finite linear span of kernels:

$$f(z) = \sum_{j=1}^M \frac{a_j}{z + w_j^*}.$$

Then, from (4.6) we get

$$f(z) = \sum_{n=1}^N e_n(z) h_n(\varphi(z)),$$

with

$$h_n(z) = \sum_{j=1}^M \frac{a_j}{z + w_j^*} e_n(w_j)^*.$$

We see that

$$\begin{aligned} \sum_{n=1}^N [h_n, h_n] &= \sum_{n=1}^N \sum_{j=1}^M e_n(w_k) \frac{a_k^* a_j}{w_k + w_j^*} e_n(w_j)^* \\ &= [f, f]. \end{aligned}$$

□

Note that neither e_j nor $f(\varphi)$ belong to $\mathbf{H}_2(\mathbb{C}_r)$. But we have:

Theorem 4.3. *The maps*

$$C_j f(z) = e_j(z) f(\varphi(z))$$

are continuous operators from the Hardy space $\mathbf{H}_2(\mathbb{C}_r)$ into itself, and

$$C_j^* \frac{1}{z + w^*} = \frac{e_j(w)^*}{z + \varphi(w)^*}.$$

In particular

$$\begin{aligned} \sum_{n=1}^N C_j C_j^* &= I \\ C_k^* C_j &= \begin{cases} I & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases} \end{aligned}$$

5. HARMONIC ANALYSIS OF REPRESENTATIONS

As in the introduction, we consider a function $k(z, w)$ positive definite on a set \mathcal{U} , and a map R from \mathcal{U} into itself. Recall that R_n was defined by (1.1). We assume that $k(z, w)$ is of the form

$$(5.1) \quad k(z, w) = 1 + t(z, w),$$

where $t(z, w)$ is positive definite in \mathcal{U} . This is equivalent to request that \mathbb{C} is contractively included in the reproducing kernel Hilbert spaces with reproducing kernel $k(z, w)$. We set

$$\Omega = \left\{ z \in \mathcal{U} ; \sum_{n=0}^{\infty} |t(R_n(z), R_n(z))| < \infty \right\}.$$

Note that this set may be empty, but that, in any case,

$$R(\Omega) \subset \Omega.$$

Lemma 5.1. *Assuming that $\Omega \neq \emptyset$. Then the infinite product*

$$(5.2) \quad K(z, w) = \prod_{n=0}^{\infty} (1 + t(R_n(z), R_n(w))), \quad z, w \in \Omega,$$

converges, and satisfies

$$(5.3) \quad K(z, w) = (1 + t(z, w))K(R(z), R(w)), \quad z, w \in \Omega,$$

Proof: Since $t(z, w)$ is positive definite in Ω we have

$$|t(R_n(z), R_n(w))| \leq \sqrt{t(R_n(z), R_n(z))} \sqrt{t(R_n(w), R_n(w))}, \quad z, w \in \Omega.$$

The Cauchy-Schwarz inequality insures that

$$\sum_{n=0}^{\infty} |t(R_n(z), R_n(w))| < \infty,$$

and so the infinite product converges. Equation (5.3) follows from the definition of the infinite product. \square

Lemma 5.2. *Assume that $\Omega \neq \emptyset$. Then, $1 \notin \mathcal{H}(t)$ and*

$$(5.4) \quad \mathcal{H}(k) = \mathbb{C} \oplus \mathcal{H}(t).$$

Proof: By hypothesis, there exists $z \in \Omega_0$ such that

$$(5.5) \quad \lim_{n \rightarrow \infty} t(R_n(z), R_n(z)) = 0.$$

Suppose that $1 \in \mathcal{H}(t)$, and let $c = \|1\|_{\mathcal{H}(t)}^2$. By formula (2.1), the kernel $t_1(z, w)$ defined by

$$t(z, w) = \frac{1}{c} + t_1(z, w)$$

is positive definite in Ω_0 . In particular we have

$$t(R_n(z), R_n(z)) \geq \frac{1}{c}, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \forall z \in \Omega_0,$$

which contradicts (5.5). From the decomposition (5.1) we then have $1 \in \mathcal{H}(k)$. Since $\mathbb{C} \cap \mathcal{H}(t) = \{0\}$ we obtain (5.4). \square

We now assume on R the following two conditions: First,

$$(5.6) \quad \forall z \in \Omega, \quad n(z) \stackrel{\text{def}}{=} \text{Card} \{ \zeta \in \Omega, \quad R(\zeta) = z \} < \infty,$$

and one of the following two conditions:

$$(5.7) \quad \forall z \in \Omega, \quad \frac{1}{n(z)} \sum_{R(\zeta)=z} e_j(\zeta) \overline{e_k(\zeta)} = \delta_{jk}, \quad \forall j, k \in J,$$

or

$$(5.8) \quad \forall z \in \Omega, \quad \frac{1}{n(z)} \sum_{R(\zeta)=z} e_j(\zeta) e_k(\zeta) = \delta_{jk}, \quad \forall j, k \in J,$$

holds.

Lemma 5.3. *Assume that (5.6) is in force. Then:*

(a) *If (5.7) is in force, the adjoint of the operator S_j is given by the formula*

$$(5.9) \quad (S_j^* f)(z) = \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega_0 \text{ such} \\ \text{that } R(\zeta)=z}} e_j(\zeta)^* f(\zeta).$$

(2) *If (5.8) is in force, the adjoint of the operator S_j is given by the formula*

$$(5.10) \quad (S_j^* f)(z) = \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} e_j(\zeta) f(\zeta).$$

Proof:

(a) Using (5.3) we write for $z, w \in \Omega_0$

$$\begin{aligned}
\frac{1}{n(z)} \sum_{R(\zeta)=z} (e_j(\zeta))^* K(\zeta, w) &= \frac{1}{n(z)} \sum_{R(\zeta)=z} (e_j(\zeta))^* (1 + t(\zeta, w)) K(R(\zeta), R(w)) \\
&= \frac{1}{n(z)} \left(\sum_{k \in J} \left(\sum_{R(\zeta)=z} (e_j(\zeta))^* e_k(\zeta) \right) e_k(w)^* \right) K(R(\zeta), R(w)) \\
&= (e_j(w))^* K(z, R(w)), \\
&= (S_j^* K_w)(z),
\end{aligned}$$

by formula (2.3), and where we have used (5.7) to go from the second to the third line. Since the kernels are dense in $\mathcal{H}(K)$ and since S_j^* is continuous, the equality extends to all $f \in \mathcal{H}(K)$.

(b) The proof is similar. One now has:

$$\begin{aligned}
\frac{1}{n(z)} \sum_{R(\zeta)=z} (e_j(\zeta)) K(\zeta, w) &= \frac{1}{n(z)} \sum_{R(\zeta)=z} (e_j(\zeta)) (1 + t(\zeta, w)) K(R(\zeta), R(w)) \\
&= \frac{1}{n(z)} \left(\sum_{k \in J} \left(\sum_{R(\zeta)=z} (e_j(\zeta)) e_k(\zeta) \right) e_k(w)^* \right) K(R(\zeta), R(w)) \\
&= (e_j(w))^* K(z, R(w)), \\
&= (S_j^* K_w)(z),
\end{aligned}$$

□

An important case where the second set of conditions hold is presented in [3]; see also Section 7 below.

Theorem 5.4. *Under hypothesis (5.6) and (5.7), or (5.6) and (5.8) the operators $(S_j)_{i \in J}$ satisfy the Cuntz relations.*

Proof of Theorem 5.4: We first suppose that (5.6) and (5.7) hold. Let $i_0, j_0 \in K$ and $f \in \mathcal{H}(K)$. We have

$$\begin{aligned}
(S_{i_0}^* S_{j_0} f)(z) &= \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} e_{i_0}(\zeta)^* (S_{j_0} f)(\zeta) \\
&= \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} e_{i_0}(\zeta)^* e_{j_0}(\zeta) f(R(\zeta)) \\
&= \left(\frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} e_{i_0}(\zeta)^* e_{j_0}(\zeta) \right) f(z) \\
&= \delta_{i_0, j_0} f(z), \quad \text{thanks to (5.7).}
\end{aligned}$$

We now assume that (5.6) and (5.8) are in force. Then,

$$\begin{aligned}
(S_{i_0}^* S_{j_0} f)(z) &= \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} e_{i_0}(\zeta) (S_{j_0} f)(\zeta) \\
&= \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} e_{i_0}(\zeta) e_{j_0}(\zeta) f(R(\zeta)) \\
&= \left(\frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega_0 \text{ such} \\ \text{that } R(\zeta)=z}} e_{i_0}(\zeta) e_{j_0}(\zeta) \right) f(z) \\
&= \delta_{i_0, j_0} f(z), \quad \text{thanks to (5.8).}
\end{aligned}$$

□

6. AN ORTHOGONAL BASIS

In this section we show that for anyone of the representations of a fixed \mathcal{O}_n in some Hilbert space \mathcal{H} , one may naturally construct an associated orthonormal basis (ONB) in \mathcal{H} . We will explore its implications for the analysis of kernel Hilbert spaces with special view to those arising from the iterated function systems in Julia set theory.

From the infinite product representation (5.2) of $K(z, w)$ we see that

$$K(z, w) = 1 + K_1(z, w),$$

where $K_1(z, w)$ is positive definite in Ω_0 . Furthermore, the function $\mathbf{1}$:

$$\mathbf{1}(z) \equiv 1, \quad z \in \Omega,$$

belongs to $\mathcal{H}(K)$, and in particular

$$e_j = S_j(\mathbf{1}) \in \mathcal{H}(K), \quad \forall j \in J.$$

In this section we wish to express K in the representation of the form (2.1) for an appropriate basis expressed in terms of the function $\mathbf{1}$ and of the S_j . We set $N = \dim \mathcal{H}(k)$, that is the cardinal of J (possibly, $N \geq \aleph_0$), and consider V the tree with at each vertex N edges associated to $(S_j)_{j \in J}$. On the vertices of the tree we have the functions

$$(6.1) \quad b_v(z) = (S_{i_0} S_{i_1} \cdots S_{i_N} \mathbf{1})(z),$$

where $N = 0, 1, 2, \dots$ and the i_j belong to the index set J , formed from an iterated application of the S_{i_j} .

Theorem 6.1. *The functions $(b_v)_{v \in V}$ form an orthonormal basis of $\mathcal{H}(K)$ and it holds that:*

$$(6.2) \quad K(z, w) = \sum_{v \in V} b_v(z) b_v(w)^*$$

where b_v is given by (6.1).

Proof: The Cuntz relations readily imply that the $(b_v)_{v \in V}$ form an orthonormal system. We need to see that it is complete. To see this it is enough to check directly that (6.2) holds. Let (3.8) be a representation of k in terms of an orthonormal basis $(e_j)_{j \in J}$ of $\mathcal{H}(k)$. The infinite product (5.2) is equal to a sum of elements of the form $f(z)f(w)^*$, where f is of the form

$$f(z) = e_{i_1}(z) e_{i_2}(R(z)) e_{i_3}(R^2(z)) \cdots e_{i_M}(R^{M-1}(z)),$$

where $M = 1, 2, \dots$ and the i_j belong to the index set J . Indeed, (5.2) is equal to the limit

$$K(z, w) = \lim_{N \rightarrow \infty} \prod_{n=0}^N \left(\sum_{j \in J} e_j(R^n(z)) (e_j(R^n(w))^*) \right), \quad z, w \in \Omega.$$

For a given N we have

$$\begin{aligned} \prod_{n=0}^N \left(\sum_{j \in J} e_j(R^n(z))(e_j(R^n(w)))^* \right) &= \\ &= \sum_{(i_1, \dots, i_N) \in J^N} e_{i_1}(z) e_{i_2}(R(z)) e_{i_3}(R^2(z)) \cdots e_{i_N}(R^N(z)), \end{aligned}$$

that is

$$\prod_{n=0}^N \left(\sum_{j \in J} e_j(R^n(z))(e_j(R^n(w)))^* \right) = \sum_{|v|=N+1} b_v(z) b_v(w)^*,$$

where we have denoted by $|v|$ the length of the path v starting at the origin. The result follows since the infinite product converges. Finally, by definition of the operators S_{i_j} we have:

$$\begin{aligned} e_{i_0}(z) e_{i_1}(R(z)) e_{i_2}(R^2(z)) \cdots e_{i_N}(R^N(z)) &= \\ &= S_{i_0} (e_{i_1}(\cdot) e_{i_2}(R(\cdot)) \cdots e_{i_N}(R^{N-1}(\cdot))) (z) \\ &= S_{i_0} (S_{i_2} (e_{i_3}(\cdot) \cdots e_{i_N}(R^{N-2}(\cdot)))) (z) \\ &\quad \vdots \\ &= (S_{i_0} S_{i_1} S_{i_2} \cdots S_{i_N}) (\mathbf{1}). \end{aligned}$$

This concludes the proof. \square

7. EXAMPLE: A JULIA SET

We consider $P(z) = z^2 - 1$ and

$$R(z) = P(P(z)) = z^4 - 2z^2.$$

We check below that the conditions (5.6) and (5.8) are in force. We first define

$$\Omega = \{w \in \mathbb{C} \text{ such that } (R_n(w))_{n \in \mathbb{N}_0} \in \ell_2\}.$$

For $z, w \in \Omega$ we set

$$(7.1) \quad K(z, w) = \prod_{n=0}^{\infty} (1 + R_n(z) R_n(w)^*).$$

Proposition 7.1. *The infinite product (7.1) converges in Ω to a function $K(z, w)$ which is positive definite there. Furthermore, K satisfies the equation*

$$K(z, w) = E(z, w) K_R(z, w), \quad z, w \in \Omega,$$

with

$$(7.2) \quad K_R(z, w) = K(R(z), R(w)) \quad \text{and} \quad E(z, w) = 1 + zw^*.$$

Proof: Since $\ell_2 \subset \ell_1$ the Cauchy-Schwarz inequality insures that

$$\sum_{n=0}^{\infty} |R_n(z)R_n(w)^*| < \infty, \quad z, w \in \Omega,$$

and so the infinite product converges there. The limit is positive definite in Ω since each of the factor is positive definite there and since a convergent product of positive definite functions is positive definite. Finally, equation (7.2) is clear from the infinite product representation of K . \square

Proposition 7.2. Ω is equal to the Fatou set at 0.

Proof: One direction is clear. If $z \in \Omega$, then $\lim_{n \rightarrow \infty} R_n(z) = 0$, and so z is in the Fatou set. Conversely, let z be in the Fatou set. Then there is n_0 such that

$$n \geq n_0 \quad \longrightarrow \quad |R_n(z)| < \frac{1}{2}.$$

But $R_{n+1}(z) = (R_n(z))^2((R_n(z))^2 - 2)$, and so

$$|R_{n+1}(z)| \leq \frac{3}{4}|R_n(z)|,$$

and $(R_n(z))_{n \in \mathbb{N}} \in \ell_2$. \square

Proposition 7.3. Let $z \in \Omega$. The equation $R(\zeta) = z$ has four solutions in Ω

Proof:

$R(\zeta) = z$ reads

$$(7.3) \quad \zeta^4 - 2\zeta^2 - z = 0,$$

and so hypothesis (5.6)

$$\forall z \in \mathbb{C}, \quad n(z) \stackrel{\text{def.}}{=} \text{Card} \{ \zeta \in \Omega_0, \quad R(\zeta) = z \} < \infty$$

holds with $n(z) = 4$ by the fundamental theorem of algebra. That the solutions belong to Ω follows from the fact that $P^{-1}(\Omega) = \Omega$ (the inverse image of the Fatou set is the Fatou set; see [6], [20]), and so $R^{-1}(\Omega) = \Omega$. \square

We note that the sums (5.8) now read, with $e_1(z) = 1$ and $e_2(z) = z$

$$\begin{aligned}
 (7.4) \quad & \sum_{R(\zeta)=z} e_j(\zeta) = 0, \\
 & \sum_{R(\zeta)=z} e_j^2(\zeta) = 4, \\
 & \sum_{R(\zeta)=z, k \neq j} e_k(\zeta) e_j(\zeta) = 0.
 \end{aligned}$$

The first one is in force because the coefficient of ζ is 0 in (7.3). The third one reduces to the first one since $e_1(z) = 1$. To verify the second one, let $x(z)$ be a complex number such that $x(z)^2 = 1 + z$. Then

$$\zeta^2 = 1 \pm x(z),$$

and the second equation follows.

$$\begin{aligned}
 (7.5) \quad & S_0 f(z) = f(R(z)) \\
 & S_1(z) = z f(R(z)).
 \end{aligned}$$

Proposition 7.4. *S_0 and S_1 are bounded operators from $\mathcal{H}(K)$ into itself. They satisfy*

$$(7.6) \quad S_0 S_0^* + S_1 S_1^* = I_{\mathcal{H}(K)}.$$

Lemma 7.5. *We have*

$$(7.7) \quad (S_0^* f)(z) = \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} f(\zeta),$$

and

$$(7.8) \quad (S_1^* f)(z) = \frac{1}{n(z)} \sum_{\substack{\zeta \in \Omega \text{ such} \\ \text{that } R(\zeta)=z}} \zeta f(\zeta)$$

Proof: We follow the argument in [3]. To prove (7.7) we write:

$$\begin{aligned}
\frac{1}{n(z)} \sum_{R(\zeta)=z} K(\zeta, w) &= \frac{1}{n(z)} \sum_{R(\zeta)=z} (1 + \zeta w^*) K(R(\zeta), R(w)) \\
&= \left(1 + \frac{\left(\sum_{R(\zeta)=z} \zeta \right) w^*}{n(z)} K(z, R(w)) \right) \\
&= K(z, R(w)), \quad \text{since (5.6) is in force} \\
&= (S_0^* K_w)(z).
\end{aligned}$$

The result follows by density since S_0^* is continuous. The argument for S_1^* is as follows:

$$\begin{aligned}
\frac{1}{n(z)} \sum_{R(\zeta)=z} \zeta K(\zeta, w) &= \frac{1}{n(z)} \sum_{R(\zeta)=z} \zeta (1 + \zeta w^*) K(R(\zeta), R(w)) \\
&= \left(1 + \frac{\left(\sum_{R(\zeta)=z} \zeta^2 \right) w^*}{n(z)} K(z, R(w)) \right) \\
&= w^* K(z, R(w)), \quad \text{since (5.8) is in force} \\
&= (S_1^* K_w)(z).
\end{aligned}$$

□

Theorem 7.6. *Assume (5.6) and (5.7) in force. Then the pair of operators (S_0, S_1) satisfies the Cuntz relations in $\mathcal{H}(K)$.*

Proof: We have

$$(S_0^* S_0 f)(z) = \frac{1}{n(z)} \sum_{R(\zeta)=z} (S_0 f)(\zeta) = \frac{1}{n(z)} \sum_{R(\zeta)=z} f(R(\zeta)) = f(z),$$

and

$$\begin{aligned}
(S_0^* S_1 f)(z) &= \frac{1}{n(z)} \sum_{R(\zeta)=z} (S_1 f)(\zeta) \\
&= \frac{1}{n(z)} \sum_{R(\zeta)=z} \zeta f(R(\zeta)) \\
&= \frac{1}{n(z)} \left(\sum_{R(\zeta)=z} \zeta \right) f(z) = 0.
\end{aligned}$$

Finally, the computation for $S_1^* S_1$ is as follows:.

$$\begin{aligned}
 (S_1^* S_1 f)(z) &= \frac{1}{n(z)} \sum_{R(\zeta)=z} \zeta (S_1 f)(\zeta) \\
 &= \frac{1}{n(z)} \sum_{R(\zeta)=z} \zeta^2 f(R(\zeta)) \\
 &= \frac{1}{n(z)} \sum_{R(\zeta)=z} \zeta^2 f(z) \\
 &= f(z),
 \end{aligned}$$

where we have used the second equality in (7.4). \square

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